IMPULSIVE TRACKING IN THE CASE OF SECOND ORDER MONOTYPE LINEAR OBJECTS

(IMPUL'SNYE PRESLEDOVANIIA V SLUCHAE LINEINYKH ODNOTIPNYKH OB'EKTOV VTOROGO PORIADKA)

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G.K. POZHARITSKII
(Moscow)

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In this paper we consider the problem [1] of the minimax of the time of encounter of two monotype controlled objects described by identical equations, with the condition that the integrals of the moduli of the controlling forces of both the tracking and the tracked objects have positive upper bounds. Such a constraint allows 'impulse' type jumps of objects on the phase plane.

In [2] we consider the problem of the tracking of monotypic linear objects with the condition that the constraints on the control resources allow only continuous motions of the objects; it is shown, that an optimum control can be designed on the basis of the optimum control which realizes a time optimal response to the point (0,0), for the linear system, coinciding with the considered system and having a resource equal to the difference between the resources of the tracking and tracked systems.

Following that idea, an optimal tracking can also be designed for the proposed problem.

1. We shall consider the problem [1] of the minimax of time T elapsing before the encounter of the tracking motion ($y_1(t)$) and the tracked motion ($z_1(t)$) described, respectively, by the equations

$$\frac{dy_1}{dt} = Ay_1 + Bu_1, \qquad \frac{dz_1}{dt} = Az_1 + Bv_1 \tag{1.1}$$

Here, y_1 and z_1 are two-dimensional vectors of the phase coordinates of the controlled objects; u_1 and v_1 are scalar controls; A is a second order square matrix and B is a two-dimensional column vector.

It is assumed, in agreement with [2], that the resources of the controls $u_1(t)$ and $v_1(t)$ which can be used when $t \geqslant \tau$ at any instant τ, are constrained by the condition

$$\int_\tau^\infty |u_1|\,dt \leqslant M_1 - \int_0^\tau |u_1|\,dt = \mu_1(\tau), \qquad M_1 > 0 \tag{1.2}$$

$$\int_{\tau}^{\infty} |v_1|\,dt \leqslant N_1 - \int_{0}^{\tau} |v_1|\,dt = \nu_1(\tau), \qquad N_1 > 0$$

We shall assume that the systems (1.1) are fully controllable and this does not affect the generality [3].

Then, there exists a linear transformation [4] which brings the system (1.1) to the form

$$y^{\cdot\cdot} + by^{\cdot} + cy = u, \qquad z^{\cdot\cdot} + bz^{\cdot} + cz = v \tag{1.3}$$

with the restrictions

$$\int_{\tau}^{\infty} |u|\,dt \leqslant M - \int_{0}^{\tau} |u|\,dt = \mu(\tau), \qquad \int_{\tau}^{\infty} |v|\,dt \leqslant N - \int_{0}^{\tau} |v|\,dt = \nu(\tau) \tag{1.4}$$

The restrictions allow controls in the form of impulse δ-functions, thus the concept of encounter needs to be clarified.

Let the points A and B have, at $t = \tau$, the same coordinates and different velocities

$$x(\tau) = y(\tau) - z(\tau) = 0, \qquad x^{\cdot}(\tau) = y^{\cdot}(\tau) - z^{\cdot}(\tau) > 0, \qquad x^{\cdot}(\tau) < \mu(\tau)$$

Here $\mu(\tau)$ is the reserve of the point A.

Let us assume that for $t = \tau$ the point B has received an impulsive control ν_1, which has lead to a velocity jumps

$$-z^{\cdot}(\tau) - \nu_1 = z^{\cdot}(\tau + 0)$$

If the control of the point A can be chosen as impulsive and such that

$$x^{\cdot}(\tau + 0)_2 = y^{\cdot}(\tau) + \mu_1(\nu_1) - z^{\cdot}(\tau) - \nu_1 = 0$$

then we shall consider that the encounter occured for $t = \tau$.

This means, that the encounter does not occur for $t = \tau$, only when the modulus of the difference of velocities satisfies the inequality

$$|y^{\cdot}(\tau) - z^{\cdot}(\tau + 0)| > \mu(\tau)$$

On the basis of such a definition of the encounter when the coordinates coincide, in order to avoid the encounter the point B must realize an impulse ν_1 satisfying the inequality

$$|y^{\cdot}(\tau) - z^{\cdot}(\tau) - \nu_1| > \mu(\tau)$$

Thus the point A can realize any admissible control

$$u\,[y(\tau),\ z(\tau),\ y^{\cdot}(\tau),\ z^{\cdot}(\tau),\ \mu(\tau),\ \nu(\tau),\ \nu_1]$$

designed on the basis of the knowledge of the positions, velocities and reserves of both points, and also of the 'intentions' ν_1 of the point B, if those intentions appear as impulsive type controls.

We shall assume that the control of the point A is designed on the basis of that information, not only in the case of coincidence of coordinates, but for all possible conditions

$$u\,[y(\tau),\ z(\tau),\ y^{\cdot}(\tau),\ z^{\cdot}(\tau),\ \mu(\tau),\ \nu(\tau),\ \nu_1]$$

concerning the control v of the point B. In that respect the point B is discriminated [3] and its control is designed only on the basis of the coordinates, velocities and the reserves of the points A and B

$$v\,[y(\tau),\ z(\tau),\ y^{\cdot}(\tau),\ z^{\cdot}(\tau),\ \mu(\tau),\ \nu(\tau)]$$

A pair of controls u° and v° and the time of encounter $T^\circ[u^\circ, v^\circ]$ will be called optimal, if for any $u \neq u^\circ$ and $v \neq v^\circ$ the inequalities

$$T\,[u, v^\circ] \geqslant T^\circ\,[u^\circ, v^\circ] \geqslant T\,[u^\circ, v]$$

are satisfied, where $T\,[u, v^\circ]$ and $T\,[u^\circ, v]$ are the times of encounter for a non-optimal u and an optimal v°, or vice versa. Introducing the notations

$$x(\tau) = y(\tau) - z(\tau), \qquad x^{\cdot}(\tau) = y^{\cdot}(\tau) - z^{\cdot}(\tau)$$

we obtain the equation

$$x^{\cdot\cdot} + bx^{\cdot} + cx = u - v \tag{1.5}$$

with the constraints (1.4) on the controls. We shall represent by $\xi(\tau)$ the difference of the reserves

$$\xi(\tau) = \mu(\tau) - \nu(\tau)$$

Following Krasovskii we shall replace, the problem of the minimax $T^\circ(u^\circ, v^\circ)$ under the conditions (1.5) and (1.4), by the problem of the time optimal response from the point $[x(\tau_0), x^{\cdot}(\tau_0)]$ to the point (0,0) for the equation

$$x^{\cdot\cdot} + bx^{\cdot} + cx = w \tag{1.6}$$

with a constraint on the control w of the form

$$\int_{\tau_0}^{\infty} |w|\,dt \leqslant \xi(\tau_0) \tag{1.7}$$

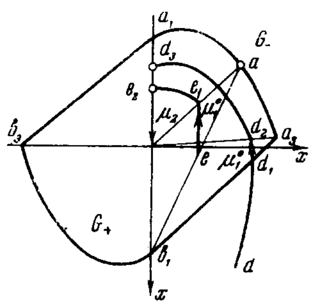

FIG. 1

2. In [4] time optimal responses from the point (0,0) to the point $(x, \dot{x})$ are considered as they appear in the solution of the problem (1.6) and (1.7). Reformulating those results for the time optimal response from the point $[x(\tau_0), \dot{x}(\tau_0)]$ to the origin of the coordinates, we shall consider their properties for various distributions of the roots of the characteristic equation $\lambda^2 + b\lambda + c = 0$ in the complex plane.

1°. *Case* $\lambda_{1,2} = \beta_{1,2} \pm i\omega,\ \beta_{1,2} < 0.$

Let us assume that initially $\xi(\tau_0) = 1$; let us direct the axis $\dot{x}$ downwards and represent on the plane $x\dot{x}$ the domain $D(t_2)$ (fig. 1), bounded by the curves $G_\pm$ and the straight lines $[a_1, b_3]$ and $[b_1, a_3]$. The curves $G_\pm$ have the equations

$$x = \pm\,\varphi_2(-s), \qquad x^{\cdot} = \pm\varphi_2^{\cdot}(-s) \qquad (0 \geqslant -s \geqslant -t_2) \tag{2.1}$$

Here $\varphi_1(t), \varphi_2(t), \varphi_1^{\cdot}(t)$, and $\varphi_2^{\cdot}(t)$ is the normal system of independent solutions of the equation $\ddot{x} + b\dot{x} + cx = 0$.

The quantity t_2 is the smallest positive root of the equation $\varphi_1(-t_2) = -1$.

The time optimal response $T^\circ(x_e, \dot{x}_e)$ from the point e whose coordinates are $(x_e, \dot{x}_e)$, located inside the domain $D(t_2)$, begins with the impulse $\mu_1^\circ < 0$ (if $x_e > 0$). The quantities μ_1° and T are determined from the equations

$$x_e + \mu_1^\circ = (+\mu_1^\circ - 1)\varphi_2^\cdot(-T^\circ), \qquad x_e = (+\mu_1^\circ - 1)\varphi_2^\cdot(-T^\circ) \quad (2.2)$$

In order to determine μ_1° and T° graphically, it is sufficient to draw the straight line b_1, e up to the point a_2 of intersection with the curve G_-, and then draw the vertical e, e_1 to the point e_1 of intersection with the line O, a_2. The segment e, e_1 is the graphical representation of the impulse μ_1. The point a_2 determines the value of the optimal time $T^\circ = s^\circ \leqslant t_2$. The segment e_2, O represents the impulse μ_2 which brings the point to the origin of the coordinates, and the curve e_1, e_2 has the equation

$$x = (\mu_1^\circ + 1)\varphi_2(-t), \qquad x^\cdot = (\mu_1^\circ + 1)\varphi_2^\cdot(-t) \qquad (-T^\circ \leqslant -t \leqslant 0)$$

If the initial point d with coordinates x, $\dot{x}$ lies outside the domain $D(t_2)$, then the point follows its integral curve d, d_1 until it intersects the boundary of the domain $D(t_2)$, then the impulse $-\mu_1 = (d_1, d_2)$ occurs, followed again by a segment of integral curve d_2, d_3 and the impulse $\mu_2 = (d_3, 0)$ which brings the point to the origin of the coordinates.

In [4] the intuitively clear property is pointed out that, when the reserve $\xi(\tau_0) \neq 1$, the optimal time and the impulse μ_1 are calculated from the expressions

$$T(x, x^\cdot, \xi) = T^\circ(x/\xi,\ x^\cdot/\xi), \qquad \mu_1(x, x^\cdot, \xi) = \xi\mu_1^\circ(x/\xi,\ x^\cdot/\xi)$$

Here $T^\circ(x, \dot{x})$ and $\mu_1^\circ(x, \dot{x})$ represent the dependence of the optimal time and the first impulse on the coordinates of the initial point, with the condition, that the 'reserve' is equal to unity.

Let us introduce new variables

$$\eta_1 = x/\xi, \qquad \eta_2 = x^\cdot/\xi$$

and let us represent the time optimal response on the $\eta_1\eta_2$ plane, assuming, as previously, that the axis η_2 is directed downwards.

The impulse $\mu_1^\circ < 0$ moves the representative point from the point $\eta_1(\tau_0), \eta_2(\tau_0)$ to the point

$$\eta_1(\tau_0 + 0) = \frac{x(\tau_0)}{\xi(\tau_0) - |\mu_1|}, \qquad \eta_2(\tau_0 + 0) = \frac{x^\cdot(\tau_0) + \mu_1}{\xi(\tau_0) - |\mu_1|}$$

Taking (2.2) into consid:ration, we get

$$\eta_1(\tau_0 + 0) = \varphi_2(-T), \qquad \eta_2(\tau_0 + 0) = \varphi_2^\cdot(-T)$$

Consequently, the time optimal response on the plane $\eta_1\eta_2$ ix represented by the broken path e, a_2, a_1, O (fig. 2).

The time optimal response e, a_2, a_1, O from a point inside the domain $D(t_2)$ begins by a jump e, a_1, followed by a segment of integral curve a_2, a_1 and ends with a jump μ_2, which brings us to the origin of the coordinates, provided that when $\xi = \dot{x} = x = 0$, $\eta_1 = \eta_2 = 0$).

If the point d lies outside the boundaries of the domain $D(t_2)$, then d d_1 is an integral curve; it is followed by a jump μ_1 which brings the point onto the curve G_-, and then after a time t_2 during which the point follows the curve there is a jump which brings the point to the origin of the coordinates.

Let us produce the tangents $[b_1, b_4)$ and $[a_1, a_4)$. Together with $[b_1, b_5)$ and $[a_1, a_5)$

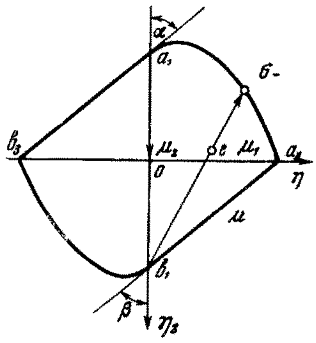

FIG. 2

which are the continuation of the axis η_2, they form two angles α and β which are utilised later. (fig. 3).

We shall now continue the graphical construction of the time optimal response, limiting ourselves to noticing the facts and referring the reader to [4] for the proof.

2°. $\lambda_{1,2} = \beta_{1,2} \pm i\omega$, $\beta_{1,2} > 0$. The time optimal solution e, a_2, a_1, O is possible only from inside the closed domain $D(t_2)$ of accessibility (fig. 3).

The curves G_+ are given by the equations (2.1) where t_2 is the smallest positive root of the equation $\varphi_1(t_2) = -1$.

If the initial point is outside $D(t_2)$ the origin of the coordinates is not accessible.

3°. $\lambda_{1,2} = \pm i\omega$ are pure imaginary. Then $t_2 = \pi/\omega$ and the domain of accessibility $D(t_2)$ is an ellipse with a unit vertical semi-axis equal to unity and the ratio of the semi-axes equal to ω (fig. 4). The time optimal response e, a_2, a_1, O is only possible from the points inside the ellipse and the origin of the coordinates is totally inaccessible from points outside the ellipse.

4°. $\lambda_2 > 0$, $\lambda_1 < 0$ are real roots of opposite signs. (fig. 5). The time optimal response and the accessibility of the origin of the coordinates is possible only for points inside the band

$$|\eta_2 \pm 1| > |\lambda_1 \eta_1|$$

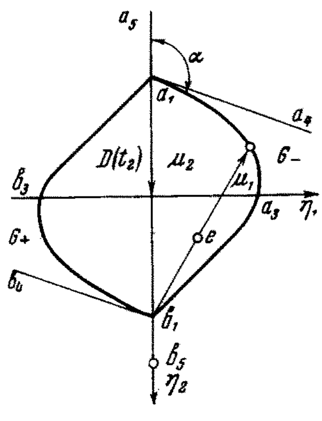

FIG. 3

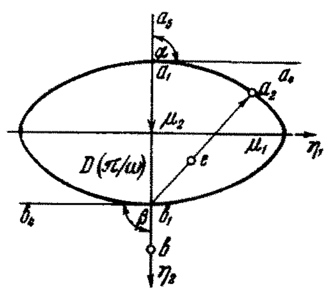

FIG. 4

Since the optimal time response $T(\eta_1, \eta_2) \to \infty$ when the initial point is taken further away from the origin inside the band, we shall call this domain $D(\infty)$.

The curve a_1, a_2 f has the equation

$$\eta_1 = -\varphi_2(-t), \qquad \eta_2 = -\varphi_2^{\cdot}(-t) \qquad (0 < t < \infty) \tag{2.4}$$

The time optimal response from a point e, located below this curve begins by a jump e, a_2 along a line b_1, e then follows the curve from a_2 to a_1 and ends with a jump to the origin of the coordinates.

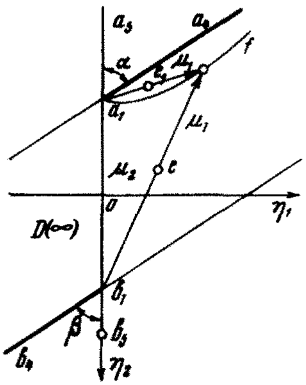

FIG. 5

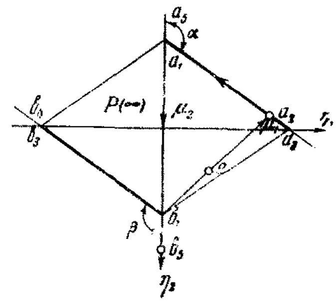

FIG. 6

The time optimal response from a point e_1, located above the curve (2.4) begins by a jump $e_1\, a_2$ along the line a_1, e_1 and then coincides with the previous response.

5°. When $\lambda_1 < 0$ and $\lambda_2 = 0$, the curve (2.4) coincides with the line $\eta_2 - 1 = -\lambda_1 \eta_1$. The rest is unchanged.

6°. When $\lambda_1 = 0$, $\lambda_2 > 0$, the domain $D(\infty)$ is a parallelogram a_1, a_3, b_1, b_3 (fig. 6).

These corresponds a finite time optimal response to each point of the parallelogram with the exception of the sides $[b_3, a_1)$ and $[a_3, b_1)$. The origin of the coordinates is inaccessible from the outside of the parallelogram or from points on those sides. The time optimal response from a point inside can be represented by the broken line e, a_2, a_1, O.

7°. When $\lambda_1 = \lambda_2 = 0$, the parallelogram $D(\infty)$ becomes the band $\eta_2 \leqslant 1$. Accessibility to the origin of the coordinates is possible from any point of the band with the exception of the half lines

$$\eta_1 = 1, \quad \eta_2 < 0; \qquad \eta_2 = -1. \quad \eta_2 > 0$$

3. We shall now proceed with the construction of the optimal controls.

3.1. Choice of the control u°. Let some state of the system

$$\eta_1(\tau) = x(\tau) \mid \xi(\tau), \qquad \eta_2(\tau) = x^{\cdot}(\tau) / \xi(\tau), \qquad \xi(\tau) > 0$$

be inside or on the border of the domain $D(t_2)$ in the cases 1, 2 and 3 and inside or on the boundary of the domains of accessibility in the cases 4, 5, 6 and 7.

We shall take for control u° the time optimal response w°, constructed taking into account the 'intentions' of the point B

$$u^\circ = w^\circ \left[\frac{x(\tau)}{\xi(\tau) + |\nu_1|}, \qquad \frac{x^{\cdot}(\tau) + \nu_1}{\xi(\tau) + |\nu_1|}\right], \qquad \nu_1 = \int_{\tau}^{\tau+0} v\, dt$$

For all the other possible cases of $\eta_1(\tau)$, $\eta_2(\tau)$, (τ) and $\xi(\tau)$, we shall assume the control u° to be identically equal to zero.

3.2. Choice of the control v°.

3.2.1. If $\xi(\tau) > 0$ and the point is not inside the angles α and β and the sides $[a_1, a_5)$ and $[b_1, b_5)$, then $v^\circ \equiv 0$.

3.2.2. If $\xi(\tau) > 0$ and the point is inside the angles α or β or on the sides $[a_1, a_5)$ and $[b_1, b_5)$, then the control v° is of impulsive nature and realizes the entire reserve of $\nu(\tau)$.

From the above, it is clear that the jump of the representative point can be directed from the position e in either of two directions $e\,a_1$ or $e\,b_1$ by choosing the sign of the impulse ν_1. In agreement with this, we shall direct the jump from the angle α or the side $[a_1, a_5)$ towards the point a_1, and the jump from the angle β or the side $[b_1, b_5)$, to the point b_1.

3.2.3. If $\xi(\tau) = 0$, we shall assume that the representative point is at infinity on the path

$$\eta_1(r) = sx(\tau), \qquad \eta_2(\tau) = sx^\circ(\lambda), \qquad 0 \leqslant s \leqslant \infty$$

If that path does not intersect the boundary line of the angles α and β, we shall assume, in agreement with 3.2.1 that $v^\circ \equiv 0$; if it does intersect the boundary line of either of those angles the control will be chosen according to 3.2.2.

3.2.4. If $\xi(\tau) < 0$, we shall take $v^\circ \equiv 0$ everywhere, except inside the angles α_1 and β_1 opposite to the angles α and β.

Inside those angles we shall take a control v° of impulsive nature and realizing the entire reserve. We shall direct the jump towards the point a_1 from the angle α_1 and towards the point b_1 from the angle β_1.

After concluding the construction of the controls u° and v° for all possible situations which might occur in the tracking process, we shall begin the proof of their optimality by a lemma.

Lemma 3.3. If for $t = \tau$ there is some state

$$x(\tau), \quad x^{\cdot}(\tau), \quad \xi(\tau)$$

located inside the angles α and β or on the sides (a_1, a_5) and (b_1, b_5) (with the exlusion of the points a_1, b_1), or if for $t = \tau$ a state $\xi(\tau) < 0$ is realized inside the angles α_1 and β_1, then as a result of the impulsive control v° given by the rules of sections 3.2.2. and 3.2.3, a state

$$x(\tau + 0), \quad x^{\cdot}(\tau + 0), \quad \xi(\tau_0 + 0)$$

is realized at the moment $\tau + 0$, such, that for $t \geqslant \tau$ there is no possible control $u\,[x(\tau),\ x^{\cdot}(\tau) + \nu(\tau),\ \xi(\tau) + |\nu(\tau)|]$ which would terminate the tracking before the time $\tau + t_2$ in the first case and which could terminate it altogether in all the other cases 2° to 7°.

Proof. We shall assume that at first $\xi(\tau) > 0$ and also, without reducing the generality, that $\eta_2(\tau) < 0$ and the point $e\,[\eta_1(\tau), \eta_2(\tau)]$ lies within the angle α. The jump $\nu(\tau)$ will bring the increments (fig. 7)

$$\Delta\eta_1 = \frac{\eta_1(\tau)\,|\,\nu(\tau)\,|}{\xi(\tau) + |\,\nu(\tau)\,|}\,, \qquad \Delta\eta_2 = \frac{1 - \eta_2(\tau)\,\mathrm{sign}\,|\,\nu(\tau)\,|}{\xi(\tau) + |\,\nu(\tau)\,|}$$

assuming $\nu(\tau) > 0$, we shall have a displacement from the point $e\,[\eta_1(\tau), \eta_2(\tau)]$ along the line

$$\frac{\eta_2 - 1}{\eta_2(\tau) - 1} = \frac{\eta_2}{\eta_2(\tau)}$$

in the direction of the line $a_1 e'$ and of magnitude

$$[\Delta\eta_1^2 + \Delta\eta_2^2]^{1/2} = \frac{|\nu(\tau)|}{\xi(\tau) + |\nu(\tau)|}(ea_1) < (ea_1)$$

i.e. after the jump, the point remains inside the angle α. Reaching the point (0.0) from the angle α requires a time known to be greater than t_2 in the first case and is impossible at all in the other cases. If $\xi(\tau) = 0$ we shall assume that the representative point is in the angle α, if the point of coordinates

$$\eta_1(\varepsilon) = x(\tau)/\varepsilon; \qquad \eta_2 = x^{\cdot}(\tau)/\varepsilon$$

is inside the angle α for a sufficiently small $\varepsilon > 0$. Splitting the impulse $\nu(\tau)$ into two: $\nu(\tau) = \varepsilon + (\nu(\tau) - \varepsilon)$, will first bring the point inside the angle α, and then make it jump to the point interior to the angle. If the point lies within the angle β_1 and $\xi(\tau) < 0$, then, assuming $\nu_1 = -\nu(\tau) < 0$, we have a jump from the point $e\,[\eta_1(\tau), \eta_2(\tau)]$ to the point e; the increments are

$$\Delta\eta_1 = \frac{\eta_1(\tau)|\nu(\tau)|}{\xi(\tau) + |\nu(\tau)|}, \qquad \Delta\eta_2 = \frac{1 + \eta_2(\tau)}{\xi(\tau) + |\nu(\tau)|}$$

along the line

$$\frac{\eta_0 + 1}{\eta_2(\tau) + 1} = \frac{\eta_2}{\eta_2(\tau)}$$

in the direction e, b_1 towards the point e'', and the magnitude of the jump satisfies the inequality

$$(\Delta\eta_1^2 + \Delta\eta_2^2)^{1/2} = \frac{|\nu(\tau)|}{\xi(\tau) + |\nu(\tau)|}(eb_1) > (eb_1)$$

Consequently, the representative point falls inside the angle β. From there the center of the coordinates can only be reached in a time greater than t_2 in the first case, and is not accessible in the other cases. This constitutes the proof.

Let us now prove the basic theorem.

Theorem 3.4. The theorem consists of four statements.

3.4.1. If the representative point has initial conditions $x(\tau_0)$, $x^{\cdot}(\tau_0)$, $\xi(\tau_0) > 0$ such that its position in the plane $\eta_1\eta_2$ is inside the domain $D(t_2)$ in the first case or inside the domains of accessibility in the cases 2 to 7 and if the tracking object keeps the optimal control

$$u^{\circ}(\tau) = w^{\circ}\left[\frac{x(\tau)}{\xi(\tau) + |\nu_1|}, \quad \frac{x^{\cdot}(\tau) + \nu_1}{\xi(\tau) + |\nu_1|}\right] \tag{3.1}$$

then for any control v, such that u° has the same sign as v, including the case $v = v^{\circ} = 0$ (at every point, except a_1, b_1), the encounter occurs at the same instant

$$\tau_0 + T(u^{\circ}, v^{\circ}) = \tau_0 + T(u^{\circ}, v^{\circ}) = \tau_0 + T^{\circ}[\eta_1(\tau_0), \eta_2(\tau_0)] \tag{3.2}$$

where $T^{\circ}[\eta_1(\tau_0), \eta_2(\tau_0)]$ is the optimal time to the origin of the coordinates for the problem

$$x^{\cdot\cdot} + bx^{\cdot} + cx = w, \qquad \int_{\tau^\circ}^{\infty} |w|\,dt \leqslant \xi(\tau_0)$$

3.4.2. If the tracking object follows the optimal control u°, and the tracked object realizes some control v, opposed in sign to the control u°, and if for this v and $u \doteq 0$ some trajectory is obtained for $\tau_1 \leqslant \tau \leqslant \tau_1 + \varepsilon$, the expenditure is

$$\int_{\tau_1}^{\tau_1+\varepsilon} |v|\,dt = \nu_\varepsilon$$

on that portion of the trajectory and the following values are obtained

$$\eta_1(\tau_1 + \varepsilon) = \frac{x(\tau_1 + \varepsilon)}{\xi(\tau_1) + |\nu_\varepsilon|}, \qquad \eta_2(\tau_1 + \varepsilon) = \frac{x^{\cdot}(\tau_1 + \varepsilon) + \nu_\varepsilon}{\xi(\tau_1) + |\nu_\varepsilon|}$$

at the end point, then the encounter occurs not later than $\tau_0 + T\,[\eta_1(\tau_0),\ \eta_2(\tau_0)]$ and the instant of encounter satisfies the inequalities

$$\tau_0 + T\,[u^\circ, v] \leqslant \tau_1 + \varepsilon + T\left[\frac{x(\tau_1+\varepsilon)}{\xi(\tau_1) + |\nu_1|},\ \frac{x^{\cdot}(\tau_1+\varepsilon) + \nu_1}{\xi(\tau_1) + |\nu_1|}\right] < \\ < \tau_0 + T^\circ[\eta_1(\tau),\ \eta_2(\tau_0)] \tag{3.3}$$

3.4.3. If the tracked object follows an optimal control v° and the tracking object deviates from the optimal law and realizes $u \neq u^\circ$, but in such a manner that the representative point always remains inside the domain $D\,(t_2)$ in the first case and inside the domains of accessibility in the cases 2 to 7, then the tracking time is evidently greater than the optimal

$$\tau_0 + T\,(u,\ v^\circ) > \tau_0 + T^\circ\,[\eta_1(\tau_0),\ \eta_2(\tau_0)] \tag{3.4}$$

and is equal to the optimal only when the control u deviates from the optimal on the set of points of the zero measure, of the optimal trajectory.

3.4.4. If the non-optimal control u drives the representative point outside the limits of the domain $D\,(t_2)$ in the first case, or satisfies the inequality $\xi(\tau_1) < 0$, and the tracked object keeps following the optimal control v°, then the encounter occurs obviously after $\tau_0 + t_2$.

$$\tau_0 + T\,(u,\ v^\circ) > \tau_0 + t_2 \tag{3.5}$$

3.4.5. If in the cases 2 to 7, the non-optimal control ≪u≫ drives the representative point outside the boundaries of the domain of accessibility, or satisfies the inequality $\xi(\tau_1) < 0$, then for an optimal $v = v^\circ$ and $\tau \geqslant \tau_1$ the encounter of the objects is altogether impossible.

We shall now proceed to the proofs of the statements of the theorem.

3.4.1. *Proof.* Let us assume that for $t = \tau_0$ the representative point is inside the domain of accessibility at the point e (fig. 7), and let us assume that at that instant the point B is taken by the impulse ν_1 into the point $e'(\tau_0)$, in agreement with the constraints $u^\circ = w^\circ$ for any $\nu_1 > 0$; at the instant $t = \tau_0 + 0$ the representative point reaches the curve G at the point $a_2\,(\tau_0 + 0)$.

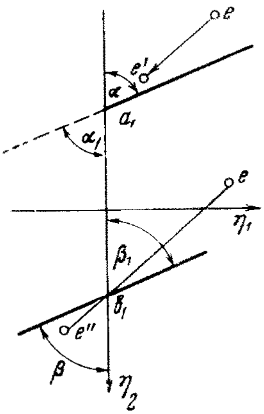

FIG. 7

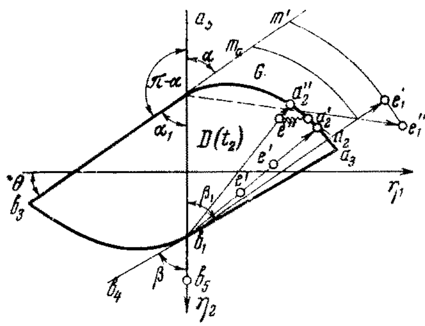

FIG. 8

Let us now assume that after $T_0 + 0$ there is a continuous or impulsive admissible control $v < 0$. This means that for any $\varepsilon > 0$

$$\int_{\tau}^{\tau+\varepsilon} v\,dt < 0$$

Since $u^\circ = w^\circ$, the inequality

$$\int_{\tau+\varepsilon}^{\tau+\varepsilon=0} u\,dt - \int_{\tau}^{\tau+\varepsilon} v\,dt = 0$$

is satisfied for arbitrarily small ε.

This means that $u^\circ - v^\circ = w^\circ$ at all points of the trajectory with the exception of the set of points of zero measure, and the encounter occurs when

$$\tau^\circ + T\,[u,\, v^\circ] = \tau^\circ + T\,[\eta_1(\tau_0),\, \eta_2(\tau_0)]$$

3.4.2. *Proof.* Let us assume, in agreement with the condition 3.4.2. that the realization of a v of sign opposite to that of w° begins when $T_1 > T_0$ and let us assume that when $u(\tau) = 0; \tau_1 \leqslant \tau \leqslant \tau_1 + \varepsilon$, and that at the instant $\tau_1 + \varepsilon$ we have the state $x(\tau_1 + \varepsilon);\ \dot{x}(\tau_1 + \varepsilon),\ \xi(\tau_1 + \varepsilon)$, represented by the point $e''(\tau_1 + \varepsilon)$ on the fig. 8. It is obvious that since w° and v have opposite signs, the control v speeds up the attainment of the axis η_2 even for $u = 0$; thus if $T_1 + \Delta T$ is the time at which the point $a_2''(T_1 + \Delta T)$ will be reached from the point $a_2'(T_1)$ when following the curve G_-, then $\tau_1 + \dot{\varepsilon}$ is evidently smaller than $T_1 + \Delta T$. Since the optimal control $w^\circ\,[\eta_1(\tau_1 + \varepsilon), \eta_2(\tau_1 + \varepsilon)]$ yields the jump e'', a_2'', then subsequently the time to reach the beginning of the measure does not exceed the time $T^\circ\,[\eta_1(\tau_1 + \varepsilon), \eta_2(\tau_1 + \varepsilon)]$. This proves the inequality (3.4.2.).

3.4.3. *Proof.* With those conditions, the representative point can remain inside the domain $D(t_2)$ or inside the domain of accessibility only if

$$\int_{\tau_0}^{T} |u|\,dt \leqslant \xi(\tau_0)$$

inside the domain $v^{\circ}=0$. This means that if u deviates from the optimal time to reach the point (0.0), then the instant of encounter satisfies the inequality

$$\tau_0 + T'[u, v^{\circ}] > \tau_0 + T^{\circ}[\eta_1(\tau_0), \eta_2(\tau_0)]$$

3.4.4. *Proof.* Let us assume that in the first case the condition $\xi(\tau_1) < 0$ is satisfied, and that the representative point is located inside the angle b_3, a_1, a_5 (fig. 8) which complements the angle α to π. It can be shown, as was done for the lemma 3.3, that any admissible jump μ_1 directed towards the point a_1 leaves the representative point inside this angle. Any continuous portion of curve located in the third quadrant inside the angle b_3, a_1, a_5, cannot lead to the realization of the equality $\eta_1 = 0$ because the signs of x and $\dot{x}$ coincide in that quadrant. Obviously a continuous portion inside the angle θ does not satisfy the equality $\eta_1 = 0$ either. Consequently, either the encounter cannot be realized at all, or the point enters inside the angle α_1. According to the lemma, the impulsive control v_0 transfers the point from the angle α_1 into the angle α and, consequently, the encounter cannot occur before t_2.

Let us assume that the state $\xi(\tau_1) > 0$ has been realized and that the point e_1 $[\eta_1(\tau_1), \eta_2(\tau_2)]$ is inside the angle b_1, a_1, a_4 and not inside the domain $D(t_2)$. As in the previous case it is easy to show that if the point remains all the time inside the angle b_5, b_1, a_3 the encounter cannot occur. If the point enters the band a_4, a_1, b_1, a_3, then any control u can only increase the distance $a_1 m$ between the point a_1 and the point m of intersection of the curve e, m with the straight line a_1, a_4.

The curve e, m is the integral line obtained for $t > \tau_1$, if for $\tau > \tau_1$, we take $u(\tau) = 0$. Fig. 8, also show the curve $e_1'm$, obtained, if at $\tau = \tau_1$, a small impulse μ_1 of the control u is applied and removed. This means, that for any control u, which leaves $\xi(\tau) > 0$, either the encounter does not occur at all or the representative point falls inside the angle α before the encounter, and an impulsive control v° leads to a situation such that the encounter can occur only after a time greater than t_2 has elapsed. If from a point inside the angle b_5, a_1, a_4 the tracking object realizes an impulse μ_1 leading to the situation $\xi(\tau_1) - |\mu_1| < 0$, then the representative point makes a jump into the left half-plane. The consequences of this jump lead to the situation $\xi(\tau_1 + 0) < 0$ which has been considered above.

3.4.5. *Proof.* The proof concerning the cases 2 to 7, when $\xi(\tau_1) < 0$ is the same as in 3.4.4. The only difference is, that, the impulse $\nu(\tau_1)$ which brings the point inside the angle α eliminates the possibility of a future encounter. If $\xi(\tau_1) > 0$ and the point e $[\eta_1(\tau_1), \eta_2(\tau_1)]$ is outside the domains of accessibility in the cases 2 and 3, then the proof repeats 3.4.4., with the same difference as far as the conclusions are concerned.

In the case 4, a point located, for $\eta_1(\tau_1)' > 0$, outside the domain of accessibility, and outside the angle α, definitly falls inside the angle b_5, b_1, a_3 and remains there for any control u which keeps $\xi(\tau) \geqslant 0$. The outsides of the domains of accessibility have analogous properties in the cases 5 to 7 when $\xi(\tau) \geqslant 0$.

This ends the proof of the theorem 3.4.

However, one should not think that the control v° given by the laws 3.2.2. and 3.2.3. is the only one in the sense of its final results, i.e. of a maximum reduction of the time necessary for the encounter.

There is obviously no better control in the cases 2 to 7. In the first case it does not appear as the best one, since it delays the encounter by the quantity t_2 which is clearly not the maximum one of all those possible. Indeed, let us assume that at the initial instant we have the following conditions

$$\mu(\tau_0) > \nu(\tau_0), \qquad x(\tau_0) = 0, \qquad \dot{x}(\tau_0) > \mu(\tau_0)$$

i.e.

$$\eta_1(\tau_0) = 0, \quad \xi(\tau_0) > 0, \quad \eta_2(\tau_0) > 1, \quad \eta_2'(\tau_0) = \dot{x}(\tau_0)/\mu(\tau_0)$$

Since $\eta_2(\tau_0) < \eta_2'(\tau_0)$, the optimal time with the reserve $\mu(\tau_0)$ will be smaller than the optimal time with the reserve $\xi(\tau_0)$, but larger than t_2 since the point $(0, \eta_2')$ is outside the domain $D(t_2)$.

$$t_2 < T^\circ[0, \eta_2'] < T^\circ[0, \eta_2(\tau_0)]$$

If, following the recommandation 3.2.2., we realize the impulse $\nu(\tau_0) > 0$, then for $\tau = \tau_0 + 0$ the conditions

$$\eta_1(\tau_0 + 0) = 0, \qquad \eta_2(\tau_0 + 0) = \eta_2'(\tau_0)$$

are met, and the optimal time is equal to $T^\circ[0, \eta_2'(\tau_0)]$. Let us reject the law 3.2.2. and choose a different tactic, namely, for $\xi(\tau) \geqslant 0$ we take $v \equiv 0$, and for $\xi(\tau) < 0$, $v = 0$ everywhere, excepth the segment of axis $\eta_1 = 0$, $0 < |\eta_2| \leqslant 1$. Inside this segment, we can take v of impulsive nature and directed towards the point b_1. For $\xi(\tau) \geqslant 0$ the encounter cannot occur earlier than at the instant

$$\tau_0 + T^\circ[0, \eta_2(\tau_0)]$$

If, on the other hand, $\xi(\tau)$ becomes negative, then the representative point cannot get inside or on the boundary of the segment $\eta_1 = 0$, $0 < |\eta_1| \leqslant 1$ in a time smaller than $T^\circ[0, \eta_2(\tau_0)]$.

However, the impulsive control $\nu(\tau)$, in that case 'throws' the representative point on the axis $\eta_1 = 0$, definitely beyond the point b_1, i.e. inside the domain $D(t_2)$ and consequently the encounter cannot occur earlier than a time t_2 after that 'throw'. To sum up, if the point B follows the proposed control v, the encounter does not occur in any case before the time

$$\tau_0 + \max\,[T^\circ[0, \eta_2(\tau_0)],\ t_2 + T^\circ[0, \eta_2'(\tau_0)]]$$

This control does not appear, generally speaking, as optimal. The construction of the optimal controls μ° and ν° in the first case, inside the domain $D(t_2)$, goes beyond the framework of the present paper.

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